Log abelian varieties : Survey (Algebraic Number Theory and Related Topics 2014)

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Log abelian varieties (Survey)

By

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Abstract

This is a survey on the theory of log abelian varieties, which is a new formulation of degenerating abelian varieties in view of log geometry in the sense of Fontaine–Illusie.

This manuscript is a survey on the theory of log abelian varieties, a joint work with T. Kajiwara and K. Kato. In Sections 0–2, we try to describe faithfully what I lectured or planned to lecture in the conference. We omit the technical details, and the description is pretty rough and even sometimes inaccurate for explanatory reasons. We include some omitted details in the last Section 3.

§ 0. Log Geometry

We explain the log geometry in the sense of Fontaine–Illusie briefly (cf. [1]).

Definition 0.1. A *log structure* on a ringed topos $(X, \mathcal{O}_X)$ is a pair of a sheaf M_X of (commutative) monoids and a homomorphism

$$\alpha\colon M_X \to \mathcal{O}_X$$

of sheaves of monoids such that

$$\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\simeq} \mathcal{O}_X^\times$$

is an isomorphism. Here we regard $\mathcal{O}_X$ as a sheaf of monoids by multiplication (not by addition).

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Example 0.2. The canonical inclusion

$$\mathcal{O}_X^\times \to \mathcal{O}_X$$

defines a log structure. We call it the *trivial log structure*.

Example 0.3. Let $\mathbf{N} = \mathbf{N}_X$ be the constant sheaf of nonnegative integers. Then the homomorphism

$$\mathcal{O}_X^\times \oplus \mathbf{N}_X \to \mathcal{O}_X$$

which is the canonical one on the first factor and which is the homomorphism sending the generator 1 of $\mathbf{N}$ to the zero section 0 in $\mathcal{O}_X$ on the second factor defines a log structure. We call it the $\mathbf{N}$-*hollow log structure*.

These examples suggest that we can regard the sheaf M_X of monoids as a kind of amplification of the sheaf $\mathcal{O}_X^\times$ of invertible sections.

An essence of log geometry is: equipped with a log structure, a singular space (or a singular object) starts to behave as if it were nonsingular.

Definition 0.4. A *log scheme* is a pair of a scheme and a log structure on its étale topos.

Practically, we mainly use the log structures satisfying a certain finiteness condition called *fs log structures* ("fs" means fine and saturated). Both the trivial log structure and the $\mathbf{N}$-hollow log structure are fs. A log scheme whose log structure is fs is called an *fs log scheme*.

Example 0.5. Let k be a field. Consider a semistable family over a trait whose closed point is $\mathrm{Spec}\,(k)$. Then the special fiber X of this family carries a natural fs log structure. In particular, the closed point itself carries a natural fs log structure, which is isomorphic to the $\mathbf{N}$-hollow one. We call this log scheme ($\mathrm{Spec}\,(k)$, $\mathbf{N}$-hollow one) the *standard log point* and denote it by s. The structure morphism of the family induces a natural morphism $X \to s$ of fs log schemes and it is log smooth (smooth in the logarithmic sense) and shares many properties with a usual smooth morphism of schemes. For example, it has an infinitesimal lifting property in the category of fs log schemes exactly in the same way as a usual smooth morphism has it in the category of schemes. Further, it has neat (Betti/ℓ-adic/p-adic) cohomologies (cohomologies in the logarithmic sense) etc.

§ 1. Log Elliptic Curves

We explain the idea of the definition of log abelian varieties in the one-dimensional case, that is, log elliptic curves.

Assume that we want to apply the log geometry explained in the previous section to the theory of abelian varieties. Our principle is the following:

Meta-conjecture. Every statement for an abelian variety can be generalized to a log abelian variety.

What should a log elliptic curve be, say, over s (the standard log point defined in Example 0.5)? A naive definition is that it is a one-dimensional proper log smooth fs log scheme over s of "genus one" in a suitable sense or that it is the special fiber of a degenerate family of elliptic curves. For example, a Tate curve with the natural log structure is such an fs log scheme: it is certainly proper and log smooth. But a problem is that it has no (natural) group structure (because the underlying singular scheme has no group structure.) It contradicts Meta-Conjecture for it is the group structure that is one of the most important features of abelian varieties.

An idea is the following:

To get a group structure, consider "the group generated by the naive one."

But in which ambient group can the naive one generate a group?

As a test case, consider the crossed two lines

$$X = \operatorname{Spec} k[x,y]/(xy)$$

endowed with the log structure charted by

$$\mathbf{N}^2 \to k[x,y]/(xy); e := \begin{bmatrix}1\\0\end{bmatrix} \mapsto x, f := \begin{bmatrix}0\\1\end{bmatrix} \mapsto y.$$

This X also has no group structure. But, before degeneration, it was

$$\operatorname{Spec} k[x,y]/(xy-q)$$

for a fixed $q \in k^\times$, which has a group structure. Indeed it represents the functor

$$(\mathrm{sch}/k) \ni T \mapsto \{(x,y) \in \mathcal{O}_T^\times \times \mathcal{O}_T^\times \mid xy = q\} = \{x \in \mathcal{O}_T^\times\},$$

i.e., $\mathbf{G}_m$. Here, (sch/k) is the category of schemes over k. (Here and hereafter, we identify the sheaf on T and its set of global sections.)

So, since to add a log structure should erase the degeneration, we can expect, in the log world, the above

$$X = \operatorname{Spec} k[x,y]/(xy) = \operatorname{Spec} k[x,y]/(xy-q) \text{ with } q = 0$$

still has a group structure in some sense. (Formally, q here is understood first as the image of 1 by $\mathbf{N} \to M_s$, and then as its image in M_X by abuse of notation. Recall that

M_s where q lives is an amplification of the sheaf of invertible sections $k^\times = \mathcal{O}_s^\times$.) In actual, X represents the functor

$$(\text{fs log sch}/s) \ni T \mapsto \{(x,y) \in M_T \times M_T \mid xy = q\} = \{x \in M_T^{\mathrm{gp}} \mid 1|x|q\}.$$

Here, s is the standard log point (Example 0.5), (fs log sch/s) denotes the category of fs log schemes over s endowed with the topology by the strict étale coverings, we give the structure morphism

$$X \to s$$

by

$$\mathbf{N} \to \mathbf{N}^2; 1 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$q \in M_T$ is the image of $q \in M_s$ by abuse of notation, M_T^{gp} is the group $\{a^{-1}b \mid a, b \in M_T\}$ associated to the monoid M_T, and, for $a, b \in M_T^{\mathrm{gp}}$, $a|b$ means that $a^{-1}b \in M_T \subset M_T^{\mathrm{gp}}$. Visually, the last set

$$\{x \in M_T^{\mathrm{gp}} \mid 1|x|q\}$$

is the bounded region $M_T \cap qM_T^{-1}$ in M_T^{gp}. Unfortunately, this subset of M_T^{gp} is not closed under multiplication. But it has a partial multiplication in the sense that for many pairs of elements of this subset, we can find its product within this subset. (For example, two elements near 1 within this subset can be multiplied within this subset.)

Here comes the main idea of definition, that is, take the subgroup

$$(T \mapsto \{x \in M_T^{\mathrm{gp}} \mid \text{there are integers } m, n \text{ such that } q^m|x|q^n\}) =: \mathbf{G}_{m,\log}^{(q)}$$

of

$$\mathbf{G}_{m,\log} := (T \mapsto M_T^{\mathrm{gp}})$$

generated by the last subset. (Precisely, $\mathbf{G}_{m,\log}^{(q)}$ is a subsheaf. Notice that we can easily see that this subsheaf $\mathbf{G}_{m,\log}^{(q)}$ is in fact a subgroup sheaf as follows: Let x, x' be sections of this subsheaf. By definition, there are integers m, n, m', n' such that

$$q^m|x|q^n, \quad q^{m'}|x'|q^{n'}.$$

Then we have

$$q^{m+m'}|xx'|q^{n+n'}.$$

Hence xx' also belongs to this subsheaf.)

We remark that the subgroup $\mathbf{G}_{m,\log}^{(q)}$ is better than $\mathbf{G}_{m,\log}$ itself in the sense that it is not too big; it is nearly representable (ind-representable). In fact we have

$$\mathbf{G}_{m,\log}^{(q)}(T) = \bigcup_{n \geq 1} \{x \in M_T^{\mathrm{gp}} \mid q^{-n}|x|q^n\}$$

(modulo sheafification), and each

$$\{x \in M_T^{\mathrm{gp}} \mid q^{-n}|x|q^n\}$$

is represented by an fs log scheme

$$\operatorname{Spec} k[x,y]/(xy - q^{2n}, q) = \operatorname{Spec} k[x,y]/(xy)$$

endowed with the log structure charted by

$$P_n := \left\langle e_n, f_n, q \,|\, e_n f_n = q^{2n} \right\rangle \to \operatorname{Spec} k[x,y]/(xy); e_n \mapsto x, f_n \mapsto y, q \mapsto 0,$$

where the monoid P_n is generated by 3 elements e_n, f_n, q with one relation $e_n f_n = q^{2n}$. This is because an x such that $q^{-n}|x|q^n$ gives the homomorphism from this monoid P_n to M_T by sending e_n to xq^n, f_n to $q^n x^{-1}$, and q to q.

Note that the underlying scheme of this fs log scheme is just the crossed two lines and independent of the index n. Let's calculate the transition morphism corresponding to the inclusion

$$\{x \in M_T^{\mathrm{gp}} \mid q^{-n}|x|q^n\} \to \{x \in M_T^{\mathrm{gp}} \mid q^{-n-1}|x|q^{n+1}\}.$$

As was said in the above, an x with $q^{-n}|x|q^n$ induces the morphism of monoids $P_n \to M_T$ sending e_n to xq^n and f_n to $q^n x^{-1}$. If we regard this x as a section of $\{x \in M_T^{\mathrm{gp}} \,|\, q^{-n-1}|x|q^{n+1}\}$, it induces the morphism of monoids $P_{n+1} \to M_T$ sending e_{n+1} to $xq^{n+1} = (xq^n)q$ and f_{n+1} to $q^{n+1}x^{-1} = q(q^n x^{-1})$. Hence, the above inclusion corresponds to the homomorphism of monoids

$$P_{n+1} \to P_n$$

sending e_{n+1} to $e_n q$ and f_{n+1} to $f_n q$. Then the induced morphism of fs log schemes is a partial log blow-down. The underlying morphism of schemes is just the zero-map, that is, the constant map into the origin, and the system looks like

$$\text{(crossed two lines)} \xrightarrow{\text{0-map}} \text{(crossed two lines)} \xrightarrow{\text{0-map}} \text{(crossed two lines)} \xrightarrow{\text{0-map}} \cdots$$

endowed with the log structures

$$P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \cdots .$$

Thus, if we only look at the underlying objects, the system is too poor. But, in actual, the transition morphisms are all monomorphisms in virtue of the existence of log structures. Here is a big difference between the usual world and the log world: In the usual world, a blow-down is not a monomorphism, but in the log world, a blow-down is a monomorphism.

A space like $\mathbf{G}_{m,\log}^{(q)}$ might be called a ghost. Formally, it belongs to the category of *algebraic log spaces*. The definition of algebraic log space is parallel to that of algebraic space. It is a functor obtained as a quotient of the sheaf represented by an fs log scheme by a log étale relation.

We summarize the story so far:

A group object appears in the limit of the successive log blow-downs.

Now we proceed to the definition of log elliptic curves. Let s be an fs log point over a separably closed field k, that is, an fs log scheme whose underlying scheme is $\operatorname{Spec} k$.

First we introduce two typical log elliptic curves over s.

Type 1. $\mathbf{G}_{m,\log}^{(q)}/q^{\mathbf{Z}}$ for a section $q \in M_s \smallsetminus k^\times$. Here the group sheaf $\mathbf{G}_{m,\log}^{(q)}$ can be defined similarly as in the case where s is a standard log point.

Type 2. A usual elliptic curve endowed with the pullback log structure from s.

Definition 1.1. A *log elliptic curve* over an fs log scheme S is a sheaf E of abelian groups on $(\text{fs log sch}/S)_{\text{ét}}$ (the subscript expresses that we endow the category with the topology by the strict étale coverings) satisfying the following conditions (1)–(3).

(1) For any $s \in S$, the pullback $E_{\overline{s}}$ of E to a geometric point over s (cf. Definition 3.6 (1)) is isomorphic to either the one of type 1 or the one of type 2.

(2) There exists a one-dimensional semiabelian scheme G over S, a section q of $M_S/\mathcal{O}_S^\times$, and an exact sequence

$$0 \to G \to E \to \mathbf{G}_{m,\log}^{(q)}/\mathbf{G}_m q^{\mathbf{Z}} \to 0$$

of sheaves of abelian groups on $(\text{fs log sch}/S)_{\text{ét}}$. Here $\mathbf{G}_{m,\log}^{(q)}$ means $\mathbf{G}_{m,\log}^{(\tilde{q})}$ for a local lift $\tilde{q} \in M_S$ of q, which is independent of choices of $\tilde{q}$ and is globally defined.

(3) (Separability.) The diagonal morphism $E \to E \times E$ is finite.

We remark that after pulling back to $\overline{s}$ ($s \in S$), the conditions (2) and (3) are deduced from the condition (1): As for (2), let $s \in S$. Then if $E_{\overline{s}}$ is of type 1, we have $G_{\overline{s}} = \mathbf{G}_m$, and the exact sequence concerned is

$$0 \to \mathbf{G}_m \to \mathbf{G}_{m,\log}^{(\tilde{q})}/\tilde{q}^{\mathbf{Z}} \to \mathbf{G}_{m,\log}^{(\tilde{q})}/\mathbf{G}_m \tilde{q}^{\mathbf{Z}} \to 0$$

for a lift $\tilde{q} \in M_{\overline{s}} \smallsetminus \mathcal{O}_{\overline{s}}^\times$ of $q_{\overline{s}}$.

If $E_{\overline{s}}$ is of type 2, we have $G_{\overline{s}} = E_{\overline{s}}$, q is invertible, and the exact sequence concerned reduces to

$$0 \to E_{\overline{s}} \to E_{\overline{s}} \to 0 \to 0.$$

Next we define level structures. In virtue of the fact that a log elliptic curve has a group structure, the definitions are exactly parallel to those in the nonlog case.

Definition 1.2. Let $N \geq 1$. Let S be an fs log scheme over $\operatorname{Spec} \mathbf{Z}[1/N]$. Let E be a log elliptic curve over S.

(1) A $\Gamma(N)$-*structure* on E is an isomorphism $(\mathbf{Z}/N\mathbf{Z})^2 \xrightarrow{\cong} \operatorname{Ker}(N \colon E \to E)$.

(2) A $\Gamma_1(N)$-*structure* on E is a section $S \to E$ of exact order N, i.e., of order N in each geometric fiber.

We define the moduli problems also just as in the nonlog case. Let $N \geq 1$.

Definition 1.3. Let S be an fs log scheme over $\operatorname{Spec} \mathbf{Z}[1/N]$. We define two functors $\overline{F}_N$ and $\overline{F}_{N,1} \colon (\text{fs log sch}/S) \to (\text{set})$ as follows. Let T be an object of $(\text{fs log sch}/S)$.

(1) $\overline{F}_N(T)$ is the set of isomorphism classes of log elliptic curves over T with $\Gamma(N)$-structure.

(2) $\overline{F}_{N,1}(T)$ is the set of isomorphism classes of log elliptic curves over T with $\Gamma_1(N)$-structure.

Then our theorem is the following.

Theorem 1.4 ([4, Theorem 5.1])**.** *Let $N \geq 3$ (resp. $N \geq 4$) and S an fs log scheme over $\operatorname{Spec} \mathbf{Z}[1/N]$ with the trivial log structure. Then $\overline{F}_N$ (resp. $\overline{F}_{N,1}$) is represented by the Deligne–Rapoport compactification $X(N)$ (resp. $X_1(N)$) over S with the log structure defined by the cusps.*

§ 2. Log Abelian Varieties

Higher dimensional theory is in progress. Here is the whole plan.

Part I (complex analytic theory [2]) 2008, J. Math. Sci. Univ. Tokyo

Part II (definition and first properties [3]) 2008, Nagoya Math. J.

Part III (case of log elliptic curves; illustration [4]) 2013, Nagoya Math. J.

Part IV (proper models [5]) 2015, Nagoya Math. J.

Part V (projective models [6]) preprint

Part VI (local moduli) in preparation

Part VII (global moduli) in preparation

Some expected applications are

– Compactification of various moduli of abelian varieties (original motivation)

– Grothendieck conjecture on monodromy coupling

– Fundamental theorem of Hodge–Arakelov theory for log elliptic curves

– Fourier–Mukai transform for log abelian varieties

We explain roughly the definition of log abelian varieties based on the definition of log elliptic curves in the previous section. See Section 3 for a precise definition.

In the definition of log elliptic curves, first we replace $\mathbf{G}_m$ with a semiabelian scheme G.

Next, over each point, we replace

$$q\colon \mathbf{Z} \to \mathbf{G}_{m,\log}$$

with a polarizable log 1-motif

$$Y \to G_{\log},$$

where Y is a locally free sheaf of $\mathbf{Z}$-modules of finite rank and $G_{\log}$ is an extension of G as a sheaf.

Then we obtain the definition of log abelian varieties. The definition of level structures and moduli problems are the same as before.

Finally, we want to explain, by an example, what kind of difficulty we encountered in this theory of log abelian varieties. As an example, consider the following statement.

Statement 1. Let S_λ be a filtered projective system of affine fs log schemes whose transition morphisms are strict. Let $S = \varprojlim S_\lambda$. Then the natural map

$$\varinjlim \{\text{log abelian variety over } S_\lambda\}_{/\cong} \to \{\text{log abelian variety over } S\}_{/\cong}$$

is bijective.

This is clearly important and should be valid. (In the above, the projective limit is represented by the projective limit of the underlying system of schemes endowed with the pullback log structure of the log structure of some S_λ.)

To show this Statement 1, take a subsheaf $A^{(\Sigma)}$ of a log abelian variety A over S, where Σ is a certain combinatorial data. This $A^{(\Sigma)}$ is called a model of A, which is an analogue of $\{x \in \mathbf{G}_{m,\log} \mid 1|x|q\}/q^{\mathbf{Z}}$ in the case of log elliptic curves, and generates the original A as a group sheaf. We descend this $A^{(\Sigma)}$ to some S_λ and it generates an abelian sheaf A_λ over S_λ.

To show that A_λ is a log abelian variety, we must check

(a) the pointwise polarizability, and

(b) the separability.

If $A^{(\Sigma)}$ is represented by an algebraic space with an fs log structure, the separability descends via $A^{(\Sigma)}$. So we encounter another problem.

Statement 2. $A^{(\Sigma)}$ is represented by an algebraic space with an fs log structure.

If the base is excellent, this Statement 2 is checked by (a log version of) Artin's criterion. The general case is reduced to this case by Statement 1..., which is a circular argument!

A solution for this issue of Statements 1 and 2 is as follows.

– We prove by a log version of Artin's criterion a variant of Statement 2 over an excellent base relaxing the condition (a) and discarding the condition (b).

– Applying this variant of Statement 2 to the above A_λ (assuming, without loss of generality, S_λ is excellent), we prove the following variant of Statement 1 by a modification of the argument before Statement 2.

Proposition 2.1 ([5, Corollary 9.3]). *Let S_λ and S be as above. Then the natural map*

$$\varinjlim \{\text{weak log abelian variety over } S_\lambda\}_{/\cong} \to \{\text{weak log abelian variety over } S\}_{/\cong}$$

is bijective.

Here a weak log abelian variety is defined by relaxing the condition of pointwise polarizability in the definition of log abelian varieties to the condition of pointwise admissibility.

– By this proposition instead of Statement 1, the above argument after Statement 2 shows the next generalization of Statement 2.

Proposition 2.2 ([5, Theorem 8.1]). *Let A be a weak log abelian variety. Then $A^{(\Sigma)}$ is represented by an algebraic space with an fs log structure.*

(We include a more precise statement of this last proposition as Proposition 3.10 below.)

This is our current best, that is, we have known yet neither if A_λ eventually satisfies the condition (a) or not, nor if Statement 1 itself is valid or not.

§ 3. Appendix: Precise Definitions

In this section, we give a precise definition of log abelian varieties, that of weak log abelian varieties, that of models, and that of algebraic log spaces. For more details, see [3, Sections 2–4], [5, Section 1], [5, Section 2], and [5, Section 10], respectively.

Let S be an fs log scheme. Let G be a commutative group scheme over the underlying scheme of S which is an extension of an abelian scheme B by a torus T. We identify G, B, and T with the sheaves on $(\text{fs log sch}/S)_{\text{ét}}$ represented by them endowed with the pullback log structure from S, respectively. (Here, the subscript "ét" means that we endow the category with the topology by the strict étale coverings.) Let

$$X := \mathcal{H}om\,(T, \mathbf{G}_m)$$

be the character group sheaf of T, where $\mathcal{H}om$ denotes the sheaf of homomorphisms. This X is a locally constant sheaf of finitely generated free $\mathbf{Z}$-modules on $(\text{fs log sch}/S)_{\text{ét}}$. Let

$$T_{\log} := \mathcal{H}om\,(X, \mathbf{G}_{m,\log}).$$

We define

$$G_{\log}$$

as the pushout of $T_{\log} \leftarrow T \to G$ in the category of sheaves of abelian groups on (fs log sch$/S)_{\text{ét}}$. Then we have the exact sequence

$$1 \to T_{\log} \to G_{\log} \to B \to 1.$$

Definition 3.1 ([3, DEFINITION 2.2]). A *log* 1-*motif* M over S consists of the following data.

– A locally constant sheaf Y of finitely generated free $\mathbf{Z}$-modules on $(\text{fs log sch}/S)_{\text{ét}}$.

– A commutative group scheme G over the underlying scheme of S which is an extension of an abelian scheme B by a torus T.

– A homomorphism $Y \to G_{\log}$.

We denote M by

$$[Y \to G_{\log}]$$

viewed as a complex of sheaves of abelian groups with Y of degree -1 and with $G_{\log}$ of degree 0.

Next, we define the dual of a log 1-motif.

Let $M = [Y \to G_{\log}]$ be a log 1-motif over S. We use the same notation as above. We define the *dual*

$$M^* = [X \to G^*_{\log}]$$

of M, where the semiabelian scheme G^* is defined as follows. Let

$$B^*$$

be the dual abelian scheme of B, and

$$T^* := \mathcal{H}om\,(Y, \mathbf{G}_m).$$

We define G^* as the sheafification of the presheaf

$$\begin{aligned}(\text{fs log sch}/S) \ni U \mapsto \{&\text{pair } (F,h) \text{ of an extension } F \text{ of } B \text{ by } \mathbf{G}_m \text{ over } U \text{ and}\\ &\text{a homomorphism } h\colon Y \to F \text{ such that the composite}\\ &Y \to F \to B \text{ coincides with } Y \to G_{\log} \to B\}_{/\cong}.\end{aligned}$$

This G^* is representable and there is a natural exact sequence

$$1 \to T^* \to G^* \to B^* \to 1.$$

The homomorphism $X \to G^*_{\log}$ is defined as follows. We can identify the sheaf $G^*_{\log}$ associated to G^* with the sheafification of the presheaf

$$\begin{aligned}(\text{fs log sch}/S) \ni U \mapsto \{&\text{pair } (F, h) \text{ of an extension } F \text{ of } B \text{ by } \mathbf{G}_m \text{ over } U \text{ and} \\ &\text{a homomorphism } h\colon Y \to F_{\log} \text{ such that the composite} \\ &Y \to F_{\log} \to B_{\log} = B \text{ coincides with } Y \to G_{\log} \to B\}_{/\cong}.\end{aligned}$$

For a section x of X, let F be the extension of B by $\mathbf{G}_m$ obtained as the pushout of $1 \to T \to G \to B \to 1$ with respect to $x\colon T \to \mathbf{G}_m$. Let h be the composite $Y \to G_{\log} \to F_{\log}$. Then the desired homomorphism $X \to G^*_{\log}$ is defined by associating to x the class of (F, h).

Next we define a polarization on a log 1-motif.

Definition 3.2 ([3, DEFINITION 2.8])**.** Let $M = [Y \to G_{\log}]$ be a log 1-motif over S. A *polarization* on M is a homomorphism

$$h\colon M \to M^* = [X \to G^*_{\log}]$$

of log 1-motifs satisfying the following four conditions (1)–(4).

(1) The induced homomorphism $B \to B^*$ is a polarization on B.

(2) The induced homomorphism $\phi\colon Y \to X$ is injective and each stalk of its cokernel is finite.

(3) For any $s \in S$ and any nontrivial $y \in Y_{\bar{s}}$, the element $\langle \phi(y), y\rangle_{\bar{s}}$ of $M^{\mathrm{gp}}_{S,\bar{s}}/\mathcal{O}^\times_{S,\bar{s}}$ is nontrivial and belongs to $M_{S,\bar{s}}/\mathcal{O}^\times_{S,\bar{s}}$. Here,

$$\langle\ ,\ \rangle\colon X \times Y \to M^{\mathrm{gp}}_S/\mathcal{O}^\times_S$$

is the canonical pairing induced by

$$Y \to G_{\log} \to G_{\log}/G \cong T_{\log}/T \cong \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m).$$

(4) The homomorphism $T_{\log} \to T^*_{\log}$ induced by $G_{\log} \to G^*_{\log}$ coincides with the one induced by ϕ.

A log 1-motif is said to be *polarizable* if there is a polarization on it.

Now we associate to a polarizable log 1-motif a log abelian variety.

Let $M = [Y \to G_{\log}]$ be a log 1-motif over S. Let the notation be as above. Define a subgroup sheaf

$$\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}$$

of $\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)$ by

$$\begin{aligned}(\text{fs log sch}/S) \ni U \mapsto \{\varphi \,|\, &\text{for any } x \in X, \text{ locally there are } y_1, y_2 \in Y \text{ such that} \\ &\langle x, y_1\rangle \text{ divides } \varphi(x) \text{ and } \varphi(x) \text{ divides } \langle x, y_2\rangle\}.\end{aligned}$$

Here, for $a, b \in M_U^{\mathrm{gp}}/\mathcal{O}_U^{\times}$, we say that a divides b if $a^{-1}b$ belongs to $M_U/\mathcal{O}_U^{\times}$.

Define a subgroup sheaf

$$G_{\log}^{(Y)}$$

of $G_{\log}$ as the inverse image of $\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}$ by the canonical homomorphism $G_{\log} \to \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)$.

The homomorphism $Y \to G_{\log}$ factors through $G_{\log}^{(Y)}$ and we denote by

$$G_{\log}^{(Y)}/Y$$

the cokernel of the induced homomorphism $Y \to G_{\log}^{(Y)}$.

If M is polarizable, then this last quotient sheaf makes an example of (polarizable) log abelian variety over S.

Example 3.3 (cf. [3, EXAMPLE 2.8.1])**.** Let s be an fs log point, that is, an fs log scheme whose underlying scheme is the spectrum of a field k. Let $X = Y = \mathbf{Z}$. Let $q \in \Gamma(s, M_s) \smallsetminus k^{\times}$. Consider the log 1-motif

$$M = [Y \to \mathbf{G}_{m,\log}; 1 \mapsto q].$$

Then the dual log 1-motif of this is

$$M^* = [X \to \mathbf{G}_{m,\log}; 1 \mapsto q]$$

and the identity map $M \to M^*$ is a polarization on M. Hence

$$A := \mathcal{H}om\,(X, \mathbf{G}_{m,\log})^{(Y)}/Y$$

is a log abelian variety. Here, the canonical pairing

$$\langle\ ,\ \rangle\colon X \times Y \to M_{s,\bar{s}}^{\mathrm{gp}}/\mathcal{O}_{s,\bar{s}}^{\times}$$

sends $(1,1)$ to the class of q, $\mathcal{H}om\,(X, \mathbf{G}_{m,\log})$ is identified with $\mathbf{G}_{m,\log}$, the subsheaf $\mathcal{H}om\,(X, \mathbf{G}_{m,\log})^{(Y)}$ is identified with

$$\mathbf{G}_{m,\log}^{(q)} := \{x \in \mathbf{G}_{m,\log} \,|\, \text{there are integers } m, n \text{ such that } q^m|x|q^n\},$$

Y is identified with the subgroup generated by q, and hence, A is identified with $\mathbf{G}_{m,\log}^{(q)}/q^{\mathbf{Z}}$.

We introduce the concept of admissible pairing.

Definition 3.4 ([3, 7.1]). Let X and Y be finitely generated free $\mathbf{Z}$-modules and let P be an fs monoid. A $\mathbf{Z}$-bilinear form $\langle\ ,\ \rangle\colon X \times Y \to P^{\mathrm{gp}}$ is *admissible* (precisely, *P-admissible*) if for any face σ of P and any homomorphism $N\colon \sigma \to \mathbf{R}_{\geq 0}$ (regarded as a monoid by addition), the pairing of $\mathbf{R}$-linear spaces

$$(X_\sigma/X_\tau)_{\mathbf{R}} \times (Y_\sigma/Y_\tau)_{\mathbf{R}} \to \mathbf{R}$$

induced by N is nondegenerate, where τ is the face $N^{-1}(0)$ of σ. Here, for a face σ of P, X_σ (resp. Y_σ) is the subgroup of X (resp. Y) consisting of all elements x (resp. y) such that $\langle x, Y\rangle$ (resp. $\langle X, y\rangle$) is contained in σ^{gp}.

Definition 3.5 ([3, 7.1]). Let X, Y be as above. Let S be an fs log scheme. A $\mathbf{Z}$-bilinear form $X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ is *admissible* if, for any $s \in S$, the induced pairing $X \times Y \to M_{S,\bar{s}}^{\mathrm{gp}}/\mathcal{O}_{S,\bar{s}}^\times$ is $M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^\times$-admissible.

Now we come to the definition of log abelian variety.

Definition 3.6 ([3, DEFINITION 4.1, 4.3]). A *log abelian variety* over an fs log scheme S is a sheaf A of abelian groups on $(\text{fs log sch}/S)_{\text{ét}}$ satisfying the following three conditions.

(1) For any $s \in S$, let $\bar{s}$ be the spectrum of a separable closure of the residue field of s endowed with the pullback log structure from S. Then there is a polarizable log 1-motif $[Y \to G_{\log}]$ over $\bar{s}$ such that the pullback of A to $(\text{fs log sch}/\bar{s})_{\text{ét}}$ is isomorphic to $G_{\log}^{(Y)}/Y$.

(2) Étale locally on S, there are a semiabelian group scheme G over S, finitely generated free $\mathbf{Z}$-modules X and Y, an admissible pairing $\langle\ ,\ \rangle\colon X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$, and an exact sequence

$$0 \to G \to A \to \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}/\overline{Y} \to 0$$

of sheaves of abelian groups, where $\overline{Y}$ is the image of Y in $\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}$.

(3) (Separability.) The diagonal morphism $A \to A \times A$ is finite.

Precisely, (3) means that for any morphism $U \to A \times A$ from a representable sheaf, the base-changed morphism $A \times_{A \times A} U \to U$ is represented by a morphism of fs log schemes whose underlying morphism of schemes is finite.

The semiabelian group scheme G in (2) in fact exists globally on S and is uniquely determined by A (see 9.2 of [3] for a proof). We define the *dimension* of A to be the relative dimension of G over S, which is a locally constant function on S.

The following proposition is proved in [4].

Proposition 3.7 ([4, PROPOSITION 1.4]). *The log elliptic curves defined in Definition* 1.1 *are one-dimensional log abelian varieties in the above sense. Conversely, one-dimensional log abelian varieties are log elliptic curves.*

Next we will give the definition of weak log abelian variety.

A log 1-motif with X and Y being constant is said to be *admissible* (resp. *nondegenerate*) if the associated canonical pairing $\langle\ ,\ \rangle\colon X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ is admissible (resp. nondegenerate). Here that the pairing is nondegenerate means that the induced two homomorphisms $X \to \mathcal{H}om(Y, M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)$ and $Y \to \mathcal{H}om(X, M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)$ are injective.

Definition 3.8 ([5, DEFINITION 1.6]). A *weak log abelian variety* over an fs log scheme S is a sheaf A of abelian groups on $(\text{fs log sch}/S)_{\text{ét}}$ satisfying the conditions (2) and (3) in Definition 3.6 of the log abelian variety together with the following condition.

(1) For any $s \in S$, there is an admissible and nondegenerate log 1-motif $[Y \to G_{\log}]$ over $\overline{s}$ such that the pullback of A to $(\text{fs log sch}/\overline{s})_{\text{ét}}$ is isomorphic to $G_{\log}^{(Y)}/Y$.

Let A be a weak log abelian variety over an fs log scheme S. We will give the definition of models of A.

By the condition (2) (cf. Definition 3.6) of the definition of a weak log abelian variety, étale locally on S, there are a semiabelian group scheme G over S, finitely generated free $\mathbf{Z}$-modules X and Y, an admissible pairing

$$\langle\ ,\ \rangle\colon X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times,$$

and an exact sequence

$$0 \to G \to A \to \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}/\overline{Y} \to 0.$$

Further étale locally on S, there are an fs monoid $\mathcal{S}$, a homomorphism of sheaves of fs monoids

$$\mathcal{S} \to M_S/\mathcal{O}_S^\times,$$

where we regard $\mathcal{S}$ as a constant sheaf, and an admissible pairing

$$X \times Y \to \mathcal{S}^{\mathrm{gp}}$$

such that the composite

$$X \times Y \to \mathcal{S}^{\mathrm{gp}} \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$$

coincides with the above $\langle\ ,\ \rangle$. We assume that the above data are given globally on the base S.

Now, let C be the submonoid of $\mathrm{Hom}\,(\mathcal{S}, \mathbf{N}) \times \mathrm{Hom}\,(X, \mathbf{Z})$ defined as

$$C = \{(N, l) \mid l(X_{N^{-1}(0)}) = 0\}.$$

Let Σ be a fan in $\mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}} \times X, \mathbf{Q})$ whose support is contained in the submonoid generated by C over $\mathbf{Q}_{\geq 0}$. Assume that Σ is stable under the action of Y on C, where $y \in Y$ acts on C by the formula

$$(N, l) \mapsto (N, l + N(\langle -, y \rangle)).$$

For any member $\Delta \in \Sigma$, let

$$\overline{V}(\Delta) \subset \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}$$

be the subsheaf defined by

$$\overline{V}(\Delta)(U) = \{\varphi \in \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)(U) \mid \mu\varphi(x) \in M_U/\mathcal{O}_U^\times \text{ for any } (\mu, x) \in \Delta^\vee\},$$

where U is an fs log scheme over S and $\Delta^\vee \subset \mathcal{S}^{\mathrm{gp}} \times X$ is the dual cone of Δ. Let

$$\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(\Sigma)}$$

be the union of such $\overline{V}(\Delta)$s for all $\Delta \in \Sigma$. Consider the image of $\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(\Sigma)}$ in $\mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}/\overline{Y}$. Then let

$$A^{(\Sigma)} \subset A$$

be the pullback of this image by the projection $A \to \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}/\overline{Y}$. We call such $A^{(\Sigma)}$ a *model* of A.

Example 3.9. Let the notation be as in Example 3.3. The canonical pairing

$$\langle\ ,\ \rangle\colon X \times Y \to M^{\mathrm{gp}}_{s,\overline{s}}/\mathcal{O}^\times_{s,\overline{s}}$$

factors as

$$X \times Y \to \mathbf{N}^{gp} \to M^{\mathrm{gp}}_{s,\overline{s}}/\mathcal{O}^\times_{s,\overline{s}},$$

where the first arrow sends $(1,1)$ to 1 and the second arrow sends 1 to the class of q. If we identify $\mathrm{Hom}\,(\mathbf{N}, \mathbf{N}) \times \mathrm{Hom}\,(X, \mathbf{Z})$ with $\mathbf{N} \times \mathbf{Z}$ naturally, the submonoid C in the above is identified with

$$\{(n, l) \in \mathbf{N} \times \mathbf{Z} \mid l = 0 \text{ if } n = 0\},$$

on which $y \in Y = \mathbf{Z}$ acts via $(n, l) \mapsto (n, l + ny)$. Let Σ be the fan consisting of all faces of the cone

$$\Delta := \{(n, l) \in \mathbf{Q}_{\geq 0} \times \mathbf{Q} \mid 0 \leq l \leq n\}$$

and its all translations. Then $\overline{V}(\Delta)$ is identified with the subsheaf

$$\{x \in \mathbf{G}_{m,\log}/\mathbf{G}_m \mid 1|x|q\}$$

of $\mathbf{G}_{m,\log}/\mathbf{G}_m$ and its pullback

$$\{x \in \mathbf{G}_{m,\log} \mid 1|x|q\}$$

to $\mathbf{G}_{m,\log}$ is represented by an fs log scheme whose underlying scheme is the crossed two lines. The model $A^{(\Sigma)}$ is the subsheaf

$$\{x \in \mathbf{G}_{m,\log} \mid \text{there is an integer } n \text{ such that } q^n|x|q^{n+1}\}/q^{\mathbf{Z}}$$

of $A = \mathbf{G}_{m,\log}^{(q)}/q^{\mathbf{Z}}$, which is represented by an fs log scheme whose underlying scheme is obtained by identifying the two points 0 and ∞ of the projective line. Some detail calculations are found in [3] 1.5.

A more precise statement of Proposition 2.2 is as follows.

Proposition 3.10 ([5, Theorem 8.1])**.** *Let A be a weak log abelian variety over an fs log scheme S. Assume that there are the above G, X, Y,*

$$\langle\ ,\ \rangle\colon X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times,$$

$$0 \to G \to A \to \mathcal{H}om\,(X, \mathbf{G}_{m,\log}/\mathbf{G}_m)^{(Y)}/\overline{Y} \to 0,$$

$\mathcal{S} \to M_S/\mathcal{O}_S^\times$, and $X \times Y \to \mathcal{S}^{\mathrm{gp}}$ globally on S. Let Σ be a Y-stable fan as above. Then the model $A^{(\Sigma)}$ is represented by a (quasi-separated) algebraic space over the underlying scheme of S endowed with an fs log structure over S.

Finally, we give a definition of log algebraic space.

Definition 3.11 ([5, 10.1])**.** Let S be an fs log scheme. A *log algebraic space* (called a log algebraic space in the second sense in [5]) over S is a sheaf F on (fs log sch$/S)_{\text{ét}}$ such that there are an fs log scheme F' over S and a surjective morphism $F' \to F$ of sheaves such that, for any fs log scheme T and any morphism $T \to F$, the fiber product $T \times_F F'$ is represented by a (quasi-separated) algebraic space with an fs log structure which is log étale over T. Here we identify an fs log scheme over S with the sheaf it represents. That $T \times_F F' \to T$ is log étale means that for any fs log scheme T' which is strict étale over $T \times_F F'$, the composite $T' \to T \times_F F' \to T$ is log étale.

A (quasi-separated) algebraic space with an fs log structure over S is a log algebraic space in the above sense ([5, Proposition 10.2]). A weak log abelian variety over S is also a log algebraic space ([5, Theorem 10.4 (1)]).

References

[1] Kato, K., Logarithmic structures of Fontaine-Illusie, *Algebraic analysis, geometry, and number theory* (J.-I. Igusa, ed.), Johns Hopkins University Press, Baltimore, 1989, 191–224.

[2] Kajiwara, T., Kato, K., and Nakayama, C., Logarithmic abelian varieties, Part I: Complex analytic theory, *J. Math. Sci. Univ. Tokyo* **15** (2008), 69–193.

[3] ———, Logarithmic abelian varieties, Part II. Algebraic theory, *Nagoya Math. J.* **189** (2008), 63–138.

[4] ———, Logarithmic abelian varieties, Part III: Logarithmic elliptic curves and modular curves, *Nagoya Math. J.* **210** (2013), 59–81.

[5] ———, Logarithmic abelian varieties, Part IV: Proper models, *Nagoya Math. J.* **219** (2015), 9–63.

[6] ———, Logarithmic abelian varieties, Part V: Projective models, preprint.